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Constrained Interpolants with Minimal $W^{k,p}$-Norm

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The purpose of the present work is to give a relatively simple treatment of the following interpolation problem. Find a function which interpolates given points and whose kth derivative is non-negative with minimal L^p-norm.

This and related problems have stimulated considerable interest in the past decade and have been solved in special cases. Without the restriction that the kth derivative be non-negative, the case $p=\infty$ is known as Favard's problem [1]. The constrained problem has been solved for $k=2$, $p=2$ by Hornung [2] and for $k=2$, $1<p\leqslant\infty$ by Iliev and Pollul [3, 4].

We treat here the case $k\geqslant 2$, $1<p<\infty$, and our approach, although traditional from the point of view of variational calculus, differs considerably from the methods used in the above-mentioned work. With our method it is also easy to treat the slightly more general case with a constraint of the form $\varphi(t)\leqslant f^{(k)}(t)\leqslant\psi(t)$ for the interpolating function f.

I. PRELIMINARIES

Let $[a,b]$ be an interval of $\mathbb{R}$ and let $t_1, t_2,..., t_{N+k}$ be points of $[a,b]$ with $a\leqslant t_1<t_2<\cdots<t_{N+k}\leqslant b$. Here N and k are integers with $N\geqslant 1$, $k\geqslant 2$. Let $y_1, y_2,..., y_{N+k}$ be real numbers. For $l=0, 1, 2,..., k$, the lth divided differences of the set of data $\{(t_i, y_i); i=1, 2,..., N+k\}$ are denoted by Δ_i^l, $i=1, 2,..., N+k-l$, and defined recursively by

$$\Delta_i^0=y_i, \qquad \Delta_i^l=\frac{l}{t_{i+l}-t_i}(\Delta_{i+1}^{l-1}-\Delta_i^{l-1}) \qquad \text{for} \quad l\geqslant 1.$$

For a continuous function f on $[a,b]$, the kth difference quotients

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$f[t_i, t_{i+1},..., t_{i+k}]$ are defined as the kth divided differences of the data $\{(t_1, f(t_1)), (t_2, f(t_2)),..., (t_{N+k}, f(t_{N+k}))\}$. It is easily seen that $f[t_i, t_{i+1},..., t_{i+k}]=0$ if f is a polynomial of degree less than k.

Let p be a real number with $1<p<\infty$ and let q be its dual exponent: $1/p+1/q=1$. $W^{k,p}(a,b)$ is the Sobolev space of measurable functions on $[a,b]$, having kth order derivatives belonging to $L^p(a,b)$, i.e., $W^{k,p}(a,b)=\{f\in C^{k-1}[a,b]; f^{(k-1)}$ is absolutely continuous, $\int_a^b |f^{(k)}(t)|^p\,dt<\infty\}$. The normalized B-splines $M_{i,k}$, $i=1,2,..., N$ are defined by

$$M_{i,k}(t)=g_{k,t}[t_i, t_{i+1},..., t_{i+k}], \qquad \text{where} \quad g_{k,t}(s)=k(s-t)_+^{k-1}$$

and a_+ is the positive part of the number a: $a_+=\max(a,0)$. Also $a_-=\max(0,-a)$ will denote the negative part of a. For each $i\in\{1,2,..., N\}$, $M_{i,k}$ is then a non-negative function with support $[t_i, t_{i+k}]$ and $\int_a^b M_{i,k}(t)\,dt=k!$

For $f\in W^{k,p}(a,b)$, the relation

$$f[x_i, x_{i+1},..., x_{i+k}]=\frac{1}{k!}\int_a^b f^{(k)}(t)\,M_{i,k}(t)\,dt, \tag{1}$$

known as Peano's theorem, now follows immediately from Taylor's theorem

$$f(s)=\sum_{\nu=0}^{k-1}\frac{f^{(\nu)}(a)}{\nu!}(s-a)^\nu+\frac{1}{k!}\int_a^b f^{(k)}(t)\,g_{k,t}(s)\,dt.$$

We introduce the class

$$F=\{f\in W^{k,p}(a,b); f^{(k)}\geqslant 0 \text{ and } f(t_i)=y_i \text{ for } i=1,2,..., N+k\}$$

and consider the problem

(P) Find $f\in F$, minimizing $\int_a^b f^{(k)}(t)^p\,dt$.

Using (1), it is easy to prove that the operator d^k/dt^k is a bijective mapping of F onto the class

$$G=\left\{g\in L^p(a,b); g\geqslant 0 \text{ and } \int_a^b g(t)\,M_{i,k}(t)\,dt=k!\,\Delta_i^k \text{ for } i=1,2,..., N\right\},$$

and thus problem (P) is equivalent to

(P′) Find $g\in G$, minimizing $\int_a^b g(t)^p\,dt$.

II. RESULTS

THEOREM 1. *Suppose that* $\Delta_i^k > 0$ *for* $i = 1, 2, ..., N$ *and that the class F is nonempty. Then problem* (P) *has a unique solution f, and there are real numbers* $\alpha_1, \alpha_2, ..., \alpha_N$ *such that*

$$f^{(k)}(t) = \left(\sum_{i=1}^{N} \alpha_i M_{i,k}(t) \right)_+^{q-1} \qquad \text{a.e. in } (a, b).$$

Moreover, there exists only one function $f \in F$ *with the property that* $f^{(k)}(t) = (\sum_{i=1}^{N} \beta_i M_{i,k}(t))_+^{q-1}$ *for some real* $\beta_1, \beta_2, ..., \beta_N$, *namely the minimizing function for problem* (P).

Before proving this theorem we state an auxiliary result, the proof of which is omitted.

LEMMA. *Let A be a measurable subset of a subinterval* (t_j, t_{j+1}) *and suppose that* $|A| > 0$. *Then the restrictions to A of the functions* $M_{i,k}$ *with* $\max(1, j+1-k) \leqslant i \leqslant \min(j, N)$ *are linearly independent on A.*

Proof of the Theorem. We consider the equivalent problem (P'). Since G is a closed, convex, and nonempty subset of the strictly convex Banach space $L^p(a, b)$, problem (P′) has a unique solution g. Put

$$E = \{t \in (a, b); g(t) > 0\}, \qquad E_i = E \cap (t_i, t_{i+k})$$

and, for $\delta > 0$,

$$E^\delta = \{t \in (a, b); g(t) > \delta\}, \qquad E_i^\delta = E^\delta \cap (t_i, t_{i+k}).$$

We then have $E_i = \bigcup_{\delta > 0} E_i^\delta$ and, since $\Delta_i^k > 0$, $|E_i| > 0$ for $i = 1, 2, ..., N$.

Let the positive number δ be so small that $|E_i^\delta| > 0$ for $i = 1, 2, ..., N$. For every $\varphi \in L^\infty(a, b)$ with $\operatorname{supp} \varphi \subset E^\delta$ and $\int_a^b \varphi(t)\, M_{i,k}(t)\, dt = 0$, $i = 1, 2, ..., N$, the function $g + \varepsilon\varphi$ belongs to G if $|\varepsilon| < \delta / \sup |\varphi|$, and thus $(d/d\varepsilon) \int_a^b (g(t) + \varepsilon\varphi(t))^p \, dt |_{\varepsilon = 0} = 0$, i.e., $\int_a^b g(t)^{p-1} \varphi(t)\, dt = 0$. This implies that the restriction of g^{p-1} to E^δ is a linear combination of the spline functions

$$g(t)^{p-1} = \sum_{i=1}^{N} \alpha_i(\delta)\, M_{i,k}(t), \qquad t \in E^\delta.$$

From the lemma it follows that the coefficients $\alpha_i(\delta)$ are uniquely determined and thus, since E^δ increases with decreasing δ, they are in fact independent of δ.

From this we conclude that

$$g(t)^{p-1} = \sum_{i=1}^{N} \alpha_i M_{i,k}(t), \qquad t \in E$$

or, equivalently,

$$g(t)=\chi_E(t)\left(\sum_{i=1}^{N}\alpha_i M_{i,k}(t)\right)_+^{q-1}, \qquad t\in(a,b)$$

where χ_E is the characteristic function of the set E.

We now want to show that the factor $\chi_E(t)$ can be dropped, or, equivalently, that $|F\setminus E|=0$, where

$$F=\left\{t\in(a,b);\ \sum_{i=1}^{N}\alpha_i M_{i,k}(t)>0\right\}.$$

Put $\psi=\chi_{F\setminus E}$. We again let δ be so small that $|E_i^\delta|>0$ for $i=1,2,\dots,N$. Then it is possible to find a function $\varphi\in L^\infty(a,b)$ with supp $\varphi\in E^\delta$ and

$$\int_{E^\delta}\varphi(t)\,M_{i,k}(t)\,dt=-\int_{F\setminus E}M_{i,k}(t)\,dt,$$

i.e.,

$$\int_a^b(\varphi(t)+\psi(t))\,M_{i,k}(t)\,dt=0, \qquad i=1,2,\dots,N.$$

Now, if $0<\varepsilon<\delta/\sup\varphi_-$, the function $g+\varepsilon(\varphi+\psi)$ belongs to G, and making the Euler variation as above with $\varepsilon\to 0+$, we obtain

$$\int_a^b g(t)^{p-1}(\varphi(t)+\psi(t))\,dt\geqslant 0,$$

i.e.,

$$\int_{E^\delta}\varphi(t)\sum_{i=1}^{N}\alpha_i M_{i,k}(t)\,dt+\int_{F\setminus E}g(t)^{p-1}\,dt\geqslant 0.$$

Now, since $g(t)=0$ on $F\setminus E$, we get

$$0\leqslant\int_{E^\delta}\varphi(t)\sum_{i=1}^{N}\alpha_i M_{i,k}(t)\,dt=-\int_{F\setminus E}\sum_{i=1}^{N}\alpha_i M_{i,k}(t)\,dt,$$

and since $\sum_{i=1}^{N}\alpha_i M_{i,k}(t)>0$ on $F\setminus E$, we conclude that $|F\setminus E|=0$, and the first part of the theorem is proved.

To prove the last statement in the theorem, suppose that $g=f^{(k)}\in G$ is given with the property

$$g(t)=\left(\sum_{i=1}^{N}\beta_i M_{i,k}(t)\right)_+^{q-1}$$

for some constants $\beta_1, \beta_2, ..., \beta_N$. Take $E = \{t; g(t) > 0\}$. Now let $h \in G$ be arbitrary. Then $\int_a^b (h(t) - g(t)) M_{i,k}(t)\, dt = 0$ for $i = 1, 2, ..., N$. Therefore $\int_a^b (h(t) - g(t)) \sum_{i=1}^N \beta_i M_{i,k}(t)\, dt = 0$, and $\int_E (h(t) - g(t)) \sum_{i=1}^N \beta_i M_{i,k}(t) dt + \int_{(a,b)\setminus E} (h(t) - g(t) \sum_{i=1}^N \beta_i M_{i,k}(t)\, dt = 0$. On E we have $g(t)^{p-1} = \sum_{i=1}^N \beta_i M_{i,k}(t)$ and outside E we have

$$h(t) - g(t) = h(t) \geqslant 0 \text{ and } \sum_{i=1}^N \beta_i M_{i,k}(t) \leqslant 0.$$

Consequently $\int_E (h(t) - g(t))\, g(t)^{p-1}\, dt \geqslant 0$, and using Hölder's inequality we obtain

$$\int_a^b g(t)^p\, dt \leqslant \int_a^b g(t)^{p-1}\, h(t)\, dt \leqslant \left(\int_a^b g(t)^p\, dt \right)^{1-(1/p)} \left(\int_a^b h(t)^p\, dt \right)^{1/p}$$

which implies that $\int_a^b g(t)^p\, dt \leqslant \int_a^b h(t)^p\, dt$, so g must be the uniquely determined minimizing function. This finishes the proof of Theorem 1.

Remark. The same technique applies if some of the Δ_i^k are allowed to be zero. In this case, if $\Delta_i^k = 0$, g must vanish on (t_i, t_{i+k}), and a representation like that in Theorem 1 is valid, where α_i is finite if $\Delta_i^k > 0$ and $\alpha_i = -\infty$ if $\Delta_i^k = 0$.

Now Theorem 1 is easily generalized in the following manner. For given measurable functions φ and ψ with $-\infty \leqslant \varphi \leqslant \psi \leqslant +\infty$ and

$$\int_a^b \varphi_+(t)\, dt < \infty, \qquad \int_a^b \psi_-(t)\, dt < \infty,$$

let us define

$$F_{\varphi,\psi} = \{ f \in W^{k,p}(a, b);\ \varphi(t) \leqslant f^{(k)}(t) \leqslant \psi(t) \text{ and}$$
$$f(t_i) = y_i \quad \text{for} \quad i = 1, 2, ..., N + k \}.$$

Consider the problem

($P_{\varphi,\psi}$) Find $f \in F_{\varphi,\psi}$ minimizing $\int_a^b |f^{(k)}(t)|^p\, dt$.

We then have

THEOREM 2. *Suppose that*

$$\int_a^b \varphi(t)\, M_{i,k}(t)\, dt < k!\, \Delta_i^k < \int_a^b \psi(t)\, M_{i,k}(t)\, dt$$

for $i = 1, 2, \ldots, N$ *and that the class* $F_{\varphi,\psi}$ *is nonempty. Then problem* $(\mathrm{P}_{\varphi,\psi})$ *has a unique solution* f, *and there are real numbers* $\alpha_1, \alpha_2, \ldots, \alpha_N$ *such that*

$$f^{(k)}(t) = \min\left\{\max\left[\left|\sum_{i=1}^{N} \alpha_i M_{i,k}(t)\right|^{q-1} \operatorname{sgn}\left(\sum_{i=1}^{N} \alpha_i M_{i,k}(t)\right), \varphi(t)\right], \psi(t)\right\}.$$

Moreover, there exists only one function $f \in F_{\varphi,\psi}$ *of this form.*

The proof, which is omitted, is obtained after only minor modifications of the previous one.

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